The synchrotron photons from the wave solution of the Dirac equation

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Abstract

The goal of this article is to show the derivation of the power spectrum of the synchrotron radiation from the Volkov solution of the Dirac equation and from S-matrix. We also generalize the Bargmann-Michel-Telegdi equation for the spin motion in case it involves the radiation term. This equation plays the crucial role in spin motion of protons in LHC and FERMILAB. The axion production in the magnetic field described by the Volkov solution is discussed.

Key words: Volkov equation, synchrotron radiation, quantum effects.

1 Introduction

Around year 1947 Floyd Haber, a young staff member and technician in the laboratory of prof. Pollock, visually observed radiation of electrons moving circularly in the magnetic field of the chamber of an accelerator (Ternov, 1994). It occurred during adjustment of cyclic accelerator-synchrotron which accelerated electrons up to 100 MeV (Elder et al.,). The radiation was observed as a bright luminous patch on the background of the chamber of the synchrotron. It was clearly visible in the daylight. In this way the "electron light" was experimentally revealed for the first time as the radiation of relativistic electrons of large centripetal acceleration. The radiation was identified with the Ivanenko and Pomeranchuk radiation, or with the Schwinger radiation and later was called the synchrotron radiation since it was observed for the first time in synchrotron. The radiation was considered as the mysterious similarly to the Roentgen mysterious x-rays.

A number of theoretical studies on the emission of a relativistic accelerating electron had been carried out long before the cited experiment. The first steps in this line was treaded by Lienard (1898). He used the Larmor formula

$$P = \frac{2}{3}\frac{e^2}{c^3}\left(\frac{d\mathbf{v}}{dt}\right)^2 = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{d\mathbf{p}}{dt}\right)^2, \tag{1}$$

and extended it to the high-velocity particles. He also received the total radiation of an electron following a circle of an circumference $2\pi R$.

In modern physics, Schwinger (1945, 1949) used the relativistic generalization of the Larmor formula to get the total synchrotron radiation. Schwinger also obtained the spectrum of the synchrotron radiation from the method which was based on the electron work on the electromagnetic field, $P = -\int(\mathbf{j}\cdot\mathbf{E})d\mathbf{x}$, where the intensity of electric field he expressed as the subtraction of the retarded and advanced electric field of a moving charge in a magnetic field, $\mathbf{E} = \frac{1}{2}(\mathbf{E}_{ret} - \mathbf{E}_{adv})$, (Schwinger, 1949).

Schott in 1907 was developing the classical theory of electromagnetic radiation of electron moving in the uniform magnetic field. His calculation was based on the Poynting vector. The goal of Schott was to explain the spectrum of radiation of atoms. Of course the theory of Schott was unsuccessful because only quantum theory is adequate to explain the emission spectrum of atoms. On the other hand the activity of Schott was not meaningless because he elaborated the theory of radiation of charged particles moving in the electromagnetic field. His theory appeared to be only of the academical interest for 40 years. Then, it was shown that the theory and specially his formula has deep physical meaning and applicability. His formula is at the present time the integral part of the every textbook on the electromagnetic field.

The classical derivation of the Schott formula is based on the Poynting vector $\mathbf{S}$ (Sokolov et al. 1966)

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}, \tag{2}$$

end $\mathbf{E}$ and $\mathbf{H}$ are intensities of the electromagnetic field of an electron moving in the constant magnetic field, where the magnetic field is in the direction of the axis z. In this case electron moves along the circle with radius R and the electromagnetic field is considered in the wave zone and in a point with the spherical coordinates r, θ, φ. In this case it is possible to show that the nonzero components of the radiated field are $-H_\theta = E_\varphi, H_\theta = E_\theta$ (Sokolov et al. 1966). They are calculated from the vector potential $\mathbf{A}$ which is expressed as the Fourier integral.

The circular classical trajectory of the electron is created by the Lorentz force $F = (e/c)(\mathbf{v}\times\mathbf{H})$. The trajectory is stationary when the radiative reaction is not considered. The radiative reaction causes the transformation of the circular trajectory to the spiral trajectory. In quantum mechanics, the trajectory is stationary when neglecting the interaction of an electron with the vacuum field. The interaction of an electron with the vacuum field, causes the electron jumps from the higher energetic level to the lower ones. In quantum electrodynamics description of the motion of electron in a homogeneous magnetic field, the stationarity of the trajectories is broken by including the mass operator into the wave equation. Then, it is possible from the mass operator to derive the power spectral formula (Schwinger, 1973). Different approach is involved in the Schwinger et al. article (1976).

It was shown that the spectral formula of the synchrotron radiation following from the quasi-classical description of the radiation of electron moving in the magnetic field is given by the following expression (Berestetzkii et al., 1989; (90.24)):

$$P(\omega) = \frac{dI}{d\omega} = -\frac{e^2 m^2 \omega}{\sqrt{\pi}\varepsilon^2}\left\{\int_x^\infty \Phi(\xi)d\xi + \left(\frac{2}{x} + \frac{\hbar\omega}{\varepsilon}\chi x^{1/2}\right)\Phi'(x)\right\}, \tag{3}$$

where

$$x = \left(\frac{mc^2}{\varepsilon}\right)^2 \left(\frac{\varepsilon\omega}{\varepsilon'\omega_0}\right)^{2/3} \stackrel{d}{=} \left(\frac{\hbar\omega}{\varepsilon'\chi}\right)^{2/3}; \quad \varepsilon = c\sqrt{p^2 + m^2c^2} \tag{4}$$

and

$$\omega_0 = \frac{v|e|H}{|\mathbf{p}|} \approx \frac{|e|H}{\varepsilon}. \tag{5}$$

is the basic frequency of circulating electron in the magnetic field. $\Phi(x)$ is so called the Airy function and it will be defined later.

Let us remark that in the classical limit i.e. with $\hbar\omega \ll \varepsilon$, or with $\varepsilon' \approx \varepsilon$, we have $x \ll 1$ and the second term in the round brackets of (3) is very small and equation reduces, after insertion of $\omega = \omega(x)$ from eq. (4) to the formula expressed in the form (Landau et al., 1988; (74.13))

$$I_l = \frac{2e^4H^2}{\sqrt{\pi}c^3m^2}\frac{mc^2}{\varepsilon}\sqrt{u}\left[-\Phi'(u) - \frac{u}{2}\int_u^\infty \Phi(u)du\right], \tag{6}$$

where

$$u = l^{2/3}\left(\frac{mc^2}{\varepsilon}\right)^2, \quad l = \frac{\omega}{\omega_0}, \tag{7}$$

and l is number of the harmonics of the circular trajectory of the electron moving in the constant magnetic field. Let us also remark that formula (6) follows also from the Schott formula if the Bessel functions of it are replaced by the Bessel functions for harmonics with $l \gg 1$.

The emitted radiation corresponds to the energy loss of electron moving in the magnetic field. According to Schwinger (1945), the energy loss is 20 eV per revolution of an electron with energy 10^8 eV and radius 0.5 m.

To calculate the total radiation from the formula (3) it is necessary to integrate over all ω from 0 to ε. However it is better to change variable using the equation (4). Using this equation, we have $\hbar\omega = \varepsilon - \varepsilon' = \varepsilon - \hbar\omega/(\chi x^{3/2})$ and from this equation it may be easy to see that

$$\hbar\omega = \left(1 - \frac{1}{1+\chi x^{3/2}}\right) = \frac{u}{1+u}; \quad u = \chi x^{3/2}. \tag{8}$$

Then, we integrate from 0 to ∞. After two integration per partes of the first term in the braces of formula (3), we get the following result (Berestetzkii et al., 1989; (90.25)):

$$I = -\frac{e^2 m^2 \chi^2}{2\sqrt{\pi}\hbar^2}\int_0^\infty \frac{4 + 5\chi x^{3/2} + 4\chi^2 x^3}{(1+\chi x^{3/2})^4}\Phi'(x) x dx =$$

$$-\frac{e^2 m^2 \chi^2}{2\sqrt{\pi}\hbar^2}\int_0^\infty \frac{4 + 5u + 4u^2}{(1+u)^4}\Phi'(z) z dz; \quad u = \chi z^{3/2}. \tag{9}$$

We will show in the next text how to determine formula (9) from the Volkov solution of the Dirac equation in the magnetic field and from the S-matrix method. From formula (9) follows also the classical expression for the synchrotron radiation.

The opening angle of radiation is not small in case of the nonrelativistic motion. The small opening angle is generated only with high energy electrons as a result of the validity of special relativity optics. According to Winick (1987), if an electron is given a total energy 5 GeV, the opening angle over which synchrotron radiation is emitted is only 0.0001 radian, or about 0.006 degree. This can be regarded as a beam with the nearly parallel rays. This is practically the same as the laser beam situation. The wave length of photons is from zero to infinity. If we want to produce maximal energy of photons at the very short length of photons, it is necessary to consider the relativistic electrons.

It it possible to consider the nonrelativistic motion of a charged particle in the strong electric and magnetic field. The trajectory is a cycloid with the very small radius and it means that the external observer sees the synchrotron radiation from the "straight line", which is perpendicular to the magnetic and electric field. The most intensive radiation is generated at the direction of "straight line" (Pardy, 2003b). The process is realized in the atmosphere of the neutron stars where the magnetic field is extremely strong.

2 The Volkov solution of the Dirac equation in the constant magnetic field

In order to derive the classical limit of the quantum radiation formula, we will suppose that the motion of the Dirac electron is performed in the following four potential:

$$A_\mu = a_\mu \varphi; \quad \varphi = kx; \quad k^2 = 0. \tag{10}$$

From equation (10), it follows that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = a_\mu k_\nu - a_\nu k_\mu = const.$, which means that electron moves in the constant electromagnetic field with the components $\mathbf{E}$ and $\mathbf{H}$. The parameters a and k can be chosen in a such a way that $\mathbf{E} = 0$. So the motion of electron is performed in the constant magnetic field.

The Volkov (1935) solution of the Dirac equation for an electron moving in a field of a plane wave is (Berestetzkii et al., 1989; Pardy, 2003a; Pardy, 2004):

$$\psi_p = \frac{u(p)}{\sqrt{2p_0}}\left[1 + e\frac{(\gamma k)(\gamma A(\varphi))}{2kp}\right]\exp\left[(i/\hbar)S\right] \tag{11}$$

and S is an classical action of an electron moving in the potential $A(\varphi)$.

$$S = -px - \int_0^{kx} \frac{e}{(kp)}\left[(pA) - \frac{e}{2}(A)^2\right] d\varphi. \tag{12}$$

It was shown that for the potential (10) the Volkov wave function is (Berestetzkii et al., 1989):

$$\psi_p = \frac{u(p)}{\sqrt{2p_0}}\left[1 + e\frac{(\gamma k)(\gamma a)}{2kp}\varphi\right] \exp\left[(i/\hbar)S\right] \tag{13}$$

with

$$S = -e\frac{ap}{2kp}\varphi^2 + e^2\frac{a^2}{6kp}\varphi^3 - px. \tag{14}$$

During the following text we will suppose that we will work in the unit system where $c = \hbar = 1$.

3 S-matrix element for photon emission

While the Larmor formula (1) involves explicitly the dependence of the radiation on the derivative of the particle velocity over the time, the quantum field theory works only with the matrix elements and power spectrum must be determined from the correct definition of the matrix element in case that electron is moving in potential (10). The alternative method can be considered and it consists in using the quantization of the Poynting vector (2) in the differential intensity as $dI = (\mathbf{r}\cdot\mathbf{S})d\Omega$. However, to our knowledge, this method was not elaborated and published. Similarly, the gravity radiation was not determined from the general relativistic definition of the Poynting vector.

The situation in the quantum physics differs from the situation in the classical one. The emission of photons by electron moving in the homogeneous magnetic field is the result of the transition of electron from the stationary state with energy E_a to stationary state with energy E_b, where $E_a > E_b$. The transition between stationary states is called spontaneous, however it is stimulated by the interaction of an electron with the virtual electromagnetic field of vacuum, or, in other words, by the interaction of electron with virtual photons. So, it is necessary to find the interaction term of an electron with vacuum photons and to solve the Dirac equation with this term and then to determine the matrix elements of the transition.

The quantum field theory expressed as the source theory was used to solved the synchrotron radiation by Schwinger (1973). In this language and methodology the original action term for the spin-0 charged particle was supplemented by the mass operator in the homogeneous magnetic field and it was shown that this mass operator involves as an integral part the power spectral formula of the synchrotron radiation. Here we use the Volkov solution of the Dirac equation and the S-matrix approach to find the probability of emission and the intensity of the synchrotron radiation. The method is nonperturbative because the Volkov solution of the Dirac equation can be expressed in the nonperturbative form.

While the Feynman diagram approach requires renormalization procedure and the Schwinger source methods requires contact terms as some form of renormalization, our method does not work with renormalization. The wave function of an electron involves the total interaction of an electron with magnetic field.

The question, if the Lorentz-Dirac equation with the radiative term can be derived from the S-matrix approach, or from the Feynman diagram approach is unanswered, and to our knowledge it was not published. On the other hand the more simple Lorentz equation for the charged particle motion in electromagnetic field was derived using the WKB approximation of the Dirac equation together with the Bargmann-Michel-Telegdi equation for the spin motion (Rafanelli and Schiller, 1964; Pardy; 1973).

It is possible to show in the quantum field theory, that the corresponding S-matrix element which describes transition from the state ψ_p to $\psi_{p'}$ with simultaneous emission of photon with polarization e' and four-momentum $k'^{\mu} = (k'_0, \mathbf{k}') = (\omega', \mathbf{k}')$ is given by the following expression (Berestetzkii et al., 1989), with $k' \to -k', S \to -S$, to be in accord with the Ritus article (Ritus, 1979):

$$M = e\int d^4x\bar{\psi}_{p'}(\gamma e'^*)\psi_p \frac{e^{-ik'x}}{\sqrt{2\omega'}}, \tag{15}$$

where ψ_p is given by the relation

$$\psi_p = \exp i\left\{e\frac{(ap)}{2(kp)}\varphi^2 - e^2\frac{a^2}{6(kp)}\varphi^3 + px\right\}\left[1 + e\frac{(\gamma k)(\gamma a)}{2(kp)}\varphi\right]\frac{u(p)}{\sqrt{2p_0}} \tag{16}$$

and $\bar{\psi}_p$ is the the conjugated function to ψ_p, or,

$$\bar{\psi}_p = \frac{\bar{u}(p)}{\sqrt{2p_0}}\left[1 + e\frac{(\gamma a)(\gamma k)}{2(kp)}\varphi\right]\exp{(i)}\left\{-e\frac{(ap)}{2(kp)}\varphi^2 + e^2\frac{a^2}{6(kp)}\varphi^3 - px\right\}. \tag{17}$$

Afer insertion of eq. (16) and (17) into eq. (15) and putting

$$\exp\left\{i\left(\frac{\alpha\varphi^2}{2} - \frac{i4\beta\varphi^3}{3}\right)\right\} = \int_{-\infty}^{\infty} ds e^{is\varphi}A(s, \alpha, \beta) \tag{18}$$

with

$$\alpha = e\left(\frac{ap}{kp} - \frac{ap'}{kp'}\right); \quad \beta = \frac{e^2a^2}{8}\left(\frac{1}{kp} - \frac{1}{kp'}\right), \tag{19}$$

we get (Ritus, 1979):

$$M = e\int_{-\infty}^{\infty}\frac{ds}{\sqrt{2\omega'}}(2\pi)^4\delta(p + sk - p' - k')\bar{u}(p') \quad \times$$

$$\left\{(\gamma e'^*)A + e\left(\frac{(\gamma a)(\gamma k)(\gamma e'^*)}{2(kp')} + \frac{(\gamma e'^*)(\gamma k)(\gamma a)}{2(kp)}\right)i\frac{\partial A}{\partial s} + \frac{e^2a^2(ke'^*)(\gamma k)}{2(kp)(kp')}\frac{\partial^2 A}{\partial s^2}\right\}u(p). \tag{20}$$

It evidently follows from eq. (18), that

$$A(s, \alpha, \beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\varphi \exp\left\{i\left(\frac{\alpha\varphi^2}{2} - \frac{4\beta\varphi^3}{3} - s\varphi\right)\right\}. \tag{21}$$

The terms $i^n\partial^n A/\partial s^n$ are the Fourier mapping of functions

$$\varphi^n \exp(i\alpha\varphi^2/2 - i4\beta\varphi^3/3). \tag{22}$$

The matrix element M is analogical to the emission of photons generated by the electron in the plane electromagnetic wave $A_\mu = a_\mu \cos(kx)$ which was proved by Ritus (1979), and the difference is in replacing the discrete s by the continual quantity. So the summation over s is replaced by the integration.

Function $A(s,\alpha,\beta)$ can be expressed by the Airy function $\Phi(y)$:

$$A(s,\alpha,\beta) = \frac{1}{\pi}(4\beta)^{-1/3}\exp\left\{-is\frac{\alpha}{8\beta} + i\frac{8\beta}{3}\left(\frac{\alpha}{8\beta}\right)^3\right\}\Phi(y), \tag{23}$$

where the Airy function $\Phi(y)$ is defined by the equation

$$\frac{d^2\Phi}{dy^2} - y\Phi = 0 \tag{24}$$

with the solution

$$\Phi(y) = \frac{1}{2}\int_{-\infty}^{\infty} du\, e^{-i\left(\frac{u^3}{3}+yu\right)} = \int_0^{\infty} du \cos\left(\frac{u^3}{3} + yu\right), \tag{25}$$

where in our case

$$y = (4\beta)^{2/3}\left[\frac{s}{4\beta} - \left(\frac{\alpha}{8\beta}\right)^2\right], \tag{26}$$

where $\beta \geq 0$. Landau et al. (1988) uses the Airy function expressed as $\Phi/\sqrt{\pi}$.

Using the formula (21) it is easy to find the differential equation for $A(s)$:

$$sA - i\alpha A' - 4\beta A'' = 0, \tag{27}$$

where $A' = \partial A/\partial s, A'' = \partial^2 A/\partial s^2$.

The evaluation of the squared matrix elements, which has physical meaning of the probability of the radiation process, involves, as can be seen, the double integral for which we use the identity:

$$\int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds' F(s)\delta(sk + p - p' - k')\delta(s'k + p - p' - k') =$$

$$\int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds' F(s)\frac{\delta(s-s')}{\delta(0)}\delta(sk + p - p' - k') =$$

$$\frac{VT}{(2\pi)^4}\int_{-\infty}^{\infty} ds \frac{F(s)}{\delta(0)}\delta(sk + p - p' - k'). \tag{28}$$

So, now, we are prepared to determine the probability of the emission of photons and we perform it in the following section.

4 Probability of emission of photons

Using the ingredients of the quantum field theory, we get for the probability of the emission of one photon an electron in unit volume per unit time (Ritus, 1979):

$$\sum_{r,r'} \frac{|M|^2}{VT} = \frac{(2\pi)^5 e^2}{\delta(0)} \int_{-\infty}^{\infty} \frac{ds}{2p_0 p_0' \omega'} \delta(sk + p - p' - k') \quad \times$$

$$\left\{ |pe''^* A - ieae''^* A'| - 2\beta(kk')(|A'|^2 + \mathrm{Re} AA''^*) \right\}, \tag{29}$$

where

$$e''_\alpha = e'_\alpha - k'_\alpha (ke')/(kk'); \quad A' = \partial A/\partial s; \quad A'' = \partial^2 A/\partial s^2. \tag{30}$$

After summation of eq. (29) over directions of polarization e' and using differential equation (27), $\sum |M|^2/VT$ will be expressed only by means of $|A|^2$ and $|A'|^2 + \mathrm{Re} AA''^*$, which can be expressed using eq. (23) in the following way:

$$|A|^2 = \frac{\Phi^2(y)}{\pi^2 (4\beta)^{2/3}}; \quad |A'|^2 + \mathrm{ReAA}''^* = \frac{y\Phi^2(y) + \Phi'^2(y)}{\pi^2 (4\beta)^{2/3}}. \tag{31}$$

Then, with

$$x = \frac{ea}{m}, \quad \chi = -\frac{kp}{m^2} x, \quad \chi' = -\frac{kp'}{m^2} x, \quad \kappa = -\frac{kk'}{m^2} x, \tag{32}$$

we have

$$\sum_{r,r'} \frac{|M|^2}{VT} = \frac{2e^2 m^2}{\delta(0) x^2 p_0 p_0' k_0'} \int_{-\infty}^{\infty} ds \left(\frac{2\chi\chi'}{\kappa} \right)^{2/3} \delta(sk + p - p' - k') \quad \times$$

$$\left\{ -\Phi^2(y) + \left(\frac{2\chi\chi'}{\kappa} \right) \left(1 + \frac{\kappa^2}{2\chi\chi'} \right)^{2/3} \left[y\Phi^2(y) + \Phi'^2(y) \right] \right\}. \tag{33}$$

In order to obtain the probability of the emission of photon by electron, it i necessary to integrate equation (33) over the final states $d^3p' d^3k' (2\pi)^{-6}$ of the electron and photon and the result divide by 1/2 and to average over polarizations of the initial electron. Integration over $\mathbf{p}'$ eliminates space δ-function and the time δ-function can be transformed into the explicit Lorentz invariant form as follows:

$$\delta(sk + p - p' - k') \frac{d^3p'}{p_0'} \quad \to \quad \frac{\delta(sk_0 + p_0 - p_0' - k_0')}{p_0'} = -\frac{\delta(s - \tilde{s})}{kp'}; \quad \tilde{s} = \frac{k'p'}{kp} \tag{34}$$

with the use of the relation

$$p_0' = \sqrt{m^2 + (s\mathbf{k} + \mathbf{p} - \mathbf{k}')^2}. \tag{35}$$

Using the equation (34) and after integration over s, we get the differential probability of the emission of photon per unit time:

$$dW = \frac{e^2 c}{4\pi^3 x \delta(0) \chi'} \left(\frac{2\chi}{u} \right)^{2/3} \quad \times$$

$$\left[-\Phi^2(y)+\left(\frac{2\chi}{u}\right)^{2/3}\left(1+\frac{u^2}{2(1+u)}\right)(y\Phi^2(y)+\Phi'^2(y))\right]\frac{d^3k'}{k'_0}, \tag{36}$$

where

$$u=\frac{\kappa}{\chi'}=\frac{kk'}{kp'}, \quad c=\frac{1}{p_0}. \tag{37}$$

The equation (36) is evidently relativistic and gauge invariant. The further properties are as follows. It does not depend on k'_1, which is the component of the photon momentum along the electric field $\mathbf{E}$. It means it does not depend on ϱ. We use further the transform

$$\frac{d^3k'}{k'_0}=\frac{xm^2\chi' u}{\chi(1+u)^2}d\varrho d\tau du. \tag{38}$$

with the obligate relation (Ritus, 1979):

$$\int_{-\infty}^{\infty}d\varrho=\delta(0). \tag{39}$$

After integration of (36) over ϱ and with regard to (39), we get the probability of emission of photons in variables u,τ without dependence of the localization of the emission the following formula:

$$dW=\frac{e^2m^2c}{2\pi^3(1+u)^2}\left(\frac{u}{2\chi}\right)^{1/3} \quad \times$$

$$\left[-\Phi^2(y)+\left(\frac{2\chi}{u}\right)^{2/3}\left(1+\frac{u^2}{2(1+u)}\right)(y\Phi^2(y)+\Phi'^2(y))\right]dud\tau. \tag{40}$$

The probability (40) has a dimension of $\mathrm{cm}^{-3}\mathrm{s}^{-1}$.

Formula (40) describes the dependence of the distribution of probability on two variables u,τ. If we use equation

$$y\Phi^2(y)+\Phi'^2(y)=\frac{1}{2}\frac{d^2}{dy^2}\Phi^2(y) \tag{41}$$

and the transformation $t=a\tau^2$; $\quad a=\left(\frac{u}{2\chi}\right)^{2/3}, d\tau=\frac{dt}{2\sqrt{at}}$, then, we get with $y=a+t$ the following result

$$\frac{dW}{du}=\frac{e^2m^2c}{2\pi^3(1+u)^2} \quad \times$$

$$\left\{-1+\left(\frac{2\chi}{u}\right)^{2/3}\left[1+\frac{u^2}{2(1+u)}\right]\frac{1}{2}\frac{d^2}{da^2}\right\}\int_0^\infty\frac{dt}{\sqrt{t}}\Phi^2(a+t); \quad a=\left(\frac{u}{2\chi}\right)^{2/3}. \tag{42}$$

Now, let us use the integral transformation (Aspnes, 1966)

$$\int_0^\infty\frac{dt}{\sqrt{t}}\Phi^2(a+t)=\frac{\pi}{2}\int_{2^{2/3}a}^\infty dy\Phi(y). \tag{43}$$

Then, we get from the formula (42)

$$\frac{dW(\chi,u)}{du} = -\frac{e^2m^2c}{4\pi^2(1+u)^2} \quad \times$$

$$\left\{\int_z^\infty dy\Phi(y) + \frac{2}{z}\left[1+\frac{u^2}{2(1+u)}\right]\Phi'(z)\right\}; \quad z = \left(\frac{u}{\chi}\right)^{2/3}. \tag{44}$$

The last formula can be written easily for small and big u as follows:

$$\frac{dW}{du} = -\Phi'(0)\frac{e^2m^2c}{2\pi^2}\left(\frac{\chi}{u}\right)^{2/3}; \quad u \ll 1, \quad \chi; \tag{45}$$

where

$$\Phi'(0) = -\frac{1}{3^{1/3}}\int_0^\infty x^{-1/3}\sin x\, dx = -\frac{1}{3^{1/3}}\int_0^\infty x^{\mu-1}\sin(ax)\, dx|_{\mu=2/3;a=1} =$$

$$-\frac{1}{3^{1/3}}\frac{\Gamma(\mu)}{a^\mu}\sin\left(\frac{\mu\pi}{2}\right)\Big|_{\mu=2/3;a=1} = -3^{1/6}\frac{\Gamma(2/3)}{2}, \tag{46}$$

$$\frac{dW}{du} = \frac{e^2m^2c}{8\pi^{3/2}u^{3/2}}\sqrt{\chi}\exp\left(\frac{-2u}{3\chi}\right); \quad u \gg 1, \quad \chi. \tag{47}$$

If we integrate the formula (44) over u and using the per partes method in the first term, we get the following formula:

$$W(\chi) = -\frac{e^2m^2c}{8\pi^2}\chi\int_0^\infty dz\frac{5+7u+5u^2}{\sqrt{z}(1+u)^3}\Phi'(z); \quad u = \chi z^{3/2}. \tag{48}$$

The last formula was derived for the first time by Goldman (1964a, 1964b) by the different way. This formula can be expressed approximately for small and big χ as follows (Ritus, 1979):

$$W(\chi) = -\frac{5e^2m^2c}{8\sqrt{3}\pi}\chi\left(1 - \frac{8\sqrt{3}}{15}\chi + ...\right); \quad \chi \ll 1, \tag{49}$$

$$W(\chi) = -\frac{7\Gamma(2/3)e^2m^2c}{54\pi}(3\chi)^{2/3}\left(1 - \frac{45}{28\Gamma(2/3)}(3\chi)^{-2/3} + ...\right); \quad \chi \gg 1. \tag{50}$$

5 Intensity of radiation

Ritus (1979) proved that the probability of radiation and the intensity of radiation differs only by the specific term beyond the integral function. So, using the Ritus proof and with regard to eq. (8) we see that the intensity of radiation can be obtained from formula (40) putting $c/p_0 \quad \to \quad 1$ and by the multiplication by the term $u(1+u)^{-1}$. We get:

$$dI = -\frac{e^2m^2}{2\pi^3}\frac{u}{(1+u)^3} \quad \times$$

$$\left(\frac{u}{2\chi}\right)^{1/3}\left\{-\Phi^2(y) + \left(\frac{2\chi}{u}\right)^{2/3}\left(1+\frac{u^2}{2(1+u)}\right)\left(y\Phi^2(y) + \Phi'^2(y)\right)\right\} du d\tau. \tag{51}$$

Then the u-distribution over intensity is of the form:

$$\frac{dI}{du} = -\frac{e^2m^2}{4\pi^2}\frac{u}{(1+u)^3}\left\{\int_z^\infty dy\Phi(y) + \frac{2}{z}\left(1+\frac{u^2}{2(1+u)}\right)\Phi'(z))\right\}; \quad z = \left(\frac{u}{\chi}\right)^{2/3}. \quad (52)$$

This formula is a quantum generalization of the classical expression for the spectral distribution of radiation of an ultrarelativistic charged particle in a magnetic field (Landau et al., 1988; (74.13)).

After integration of (52) over u and using the per partes method in the first term, we get the formula of the total radiation of photons by electron in the constant magnetic field. The formula is as follows (Ritus, 1979):

$$I = -\frac{e^2m^2}{2\sqrt{\pi}\hbar^2}\chi^2\int_0^\infty dzz\frac{4+5u+4u^2}{2(1+u)^4}\Phi'(z); \quad u = \chi z^{3/2}. \quad (53)$$

This formula can be transformed with $u = \chi x^{3/2}$ to the following one:

$$I = -\frac{e^2m^2\chi^2}{2\sqrt{\pi}\hbar^2}\int_0^\infty \frac{4+5\chi x^{3/2}+4\chi^2x^3}{(1+\chi x^{3/2})^4}\Phi'(x)xdx. \quad (54)$$

The formula (54) is equivalent with the formula (3) and the formula (52) is identical with the formula (3). There is no doubt that the Schott formula can be derived by the formalism used in this text.

6 Discussion

We have seen how to get the quantum description of the synchrotron radiation from the Volkov solution of the Dirac equation and from the formalism of the relativistic quantum theory of radiation. At the same time we have shown that the quantum synchrotron radiation leads to the classical synchrotron radiation in the classical limit.

The synchrotron radiation evidently influences the motion of the electron in accelerators. The corresponding equation which describes the classical motion is so called the Lorentz-Dirac equation, which differs from the the so called Lorentz equation only by the additional term which describes the radiative corrections. The equation with the radiative term is as follows (Landau et al., 1988):

$$m\frac{dv_\mu}{ds} = \frac{e}{c}F_{\mu\nu}v^\nu + g_\mu, \quad (55)$$

where the radiative term was derived by Landau et al. in the form (Landau et al., 1988)

$$g_\mu = \frac{2e^3}{3mc^3}\frac{\partial F_{\mu\nu}}{\partial x^\alpha}v^\nu v^\alpha - \frac{2e^4}{3m^2c^5}F_{\mu\alpha}F^{\beta\alpha}v_\beta + \frac{2e^4}{3m^2c^5}\left(F_{\alpha\beta}v^\beta\right)\left(F^{\alpha\gamma}v_\gamma\right)v_\mu. \quad (56)$$

Bargmann, Michel and Telegdi (Berestetzkii, 1989;) derived so called BMT equation for motion of spin in the electromagnetic field, in the form

$$\frac{da_\mu}{ds} = 2\mu F_{\mu\nu}a^\nu - 2\mu' v_\mu F^{\nu\lambda}v_\nu a_\lambda, \quad (57)$$

where a_μ is so called axial vector describing the classical spin. It was shown by Rafanelli and Schiller (1964), (Pardy, 1973) that this equation can be derived from the classical limit, i.e. from the WKB solution of the Dirac equation with the anomalous magnetic moment.

It is meaningful to consider the BMT equation with the radiative corrections to express the influence of the synchrotron radiation on the motion of spin. To our knowledge such equation, the generalized BMT equation, was not published and we here present the conjecture of the form of such equation. The equation is of the following form:

$$\frac{da_\mu}{ds} = 2\mu F_{\mu\nu}a^\nu - 2\mu' v_\mu F^{\nu\lambda}v_\nu a_\lambda + g_{(axial)\mu}, \tag{58}$$

where the term $g_{(axial)\mu}$ is generated as the "axialization"' of the radiation term g_μ. Or,

$$g_\mu = \frac{2e^3}{3mc^3}\frac{\partial F_{\mu\nu}}{\partial x^\alpha}v^\nu a^\alpha - \frac{2e^4}{3m^2c^5}F_{\mu\alpha}F^{\beta\alpha}a_\beta + \frac{2e^4}{3m^2c^5}\left(F_{\alpha\beta}v^\beta\right)\left(F^{\alpha\gamma}v_\gamma\right)a_\mu. \tag{59}$$

We are aware that the axialization is not unambiguous and it is evident, that it can be submitted for theoretical investigation. The future physics will give the answer if the equation corresponds to physical reality. Such equation will have fundamental meaning for the work of LHC where the synchrotron radiation influences the spin motion of protons in LHC.

The formalism used in case of the synchrotron radiation can be also applied in the situation where the axion is produced in the magnetic field. Axion was introduced by Peccei and Quinn (1977) as the pseudoscalar particle. It was introduced as the logical necessity of the correct physical theory and it means that there is the great probability that axions will be detected for instance during the experiments on LHC.

One of the corresponding Lagrangian describing the interaction of the axion field a with the electron field ψ is as follows (Skobelev, 1997):

$$\mathcal{L} = -ic\left(\frac{m_a}{f}\right)a(\bar{\psi}\gamma^5\psi), \tag{60}$$

where f is related to the coupling constant.

According to Skobelev (1997) the intensity if emission of axions by the electron moving in the constant electromagnetic magnetic field can be approximated by two formulas which follows from the general theory.

$$I_a = \frac{g^2m^2}{\pi}\chi^4; \quad g = \frac{cm}{f}; \quad \chi \ll 1; \quad \frac{m_a}{m} \ll \chi, \tag{61}$$

and/or

$$I_a = \frac{7\Gamma(2/3)g^2m^2}{2\pi 3^{13/3}}\chi^{2/3}; \quad \chi \gg 1; \quad \frac{m_a}{m} \ll \chi. \tag{62}$$

Axion is used also to explain the absence of the electrical dipole moment of the neutron. Axion is chargeless, spinless and interact with the ordinary matter only very weakly. If it is not confined, then the following decay equation is valid:

$$n \to e + p + \bar{\nu}_e + a, \tag{63}$$

which can be verified in experiment as the proof of the existence of axion in a sense that axion can decay into neutrinos as follows

$$a \to \nu_e + \bar{\nu}_e. \tag{64}$$

On the other hand if we use the plasma of particles $e, p, \bar{\nu}_e, a$, then the inversion equation to (88) is valid:

$$e + p + \bar{\nu}_e + a \to n \tag{65}$$

and this equation can be also used as the proof of the existence of axion. If we prepare the same plasma without axions, then no neutron will be generated. It seems that these simple experiments can be considered as a crucial ones for the proof of the existence of axions.

The decay of neutron and axion can be considered and calculated in the electromagnetic field as was shown by Skobelev (1997; 1999).

Khalilov et al. (1995) calculated production of the of W^- and Z^0 bosons by electron in the intense electromagnetic field. For the first process they used the following matrix element

$$M_{e\to W} = -i\frac{g}{2\sqrt{2}}\int d^4x\bar{\psi}_\nu(1-\gamma^5)\gamma^\mu\psi_e\phi_\mu, \tag{66}$$

where $\psi_\nu, \psi_e, \phi_\mu$ are wave functions of neutrino, electron and W-boson.

In case for the production of the Z-boson Khalilov et al. used the following matrix element

$$M_{e\to Z} = \tilde{g}\int d^4x\bar{\psi}_e\gamma^\lambda(g_V + g_A\gamma^5)\psi_e Z_\lambda. \tag{67}$$

It has been calculated the probability of creation and the total cross-section to every process.

It is evident that all interaction of particle physics occurring in the accelerators and LHC can be immersed into the intense electromagnetic field of laser, or laser pulse or magnetic field. The theoretical investigation can then be performed using the Volkov solution and the S-matrix method. This will obviously become the integral part of the future physics of elementary particles.

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